Absolute value, fractional part and Quadratic Trigonometry.

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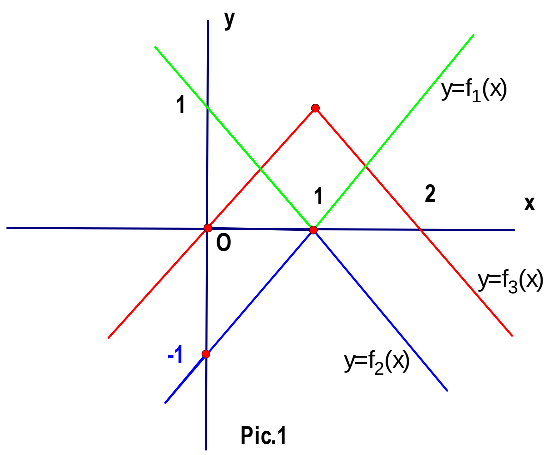
Abstract

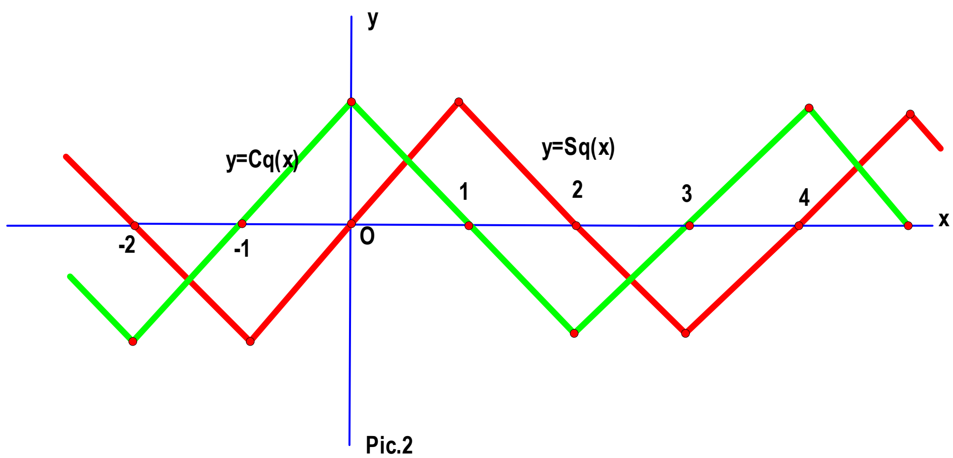
In that short note we will consider quadratic analogs of functions $\sin x$ and $\cos x$ which unlike of the latter can be defined constructively as combination absolute value function $|x|$ and fractional part function $\{x\}$. Thus, long before the study of trigonometric functions, it is possible to construct and study their quadratic constructive analogues and at the same time, presenting in the synthesis such topics as transformation of graphs, composition of functions, periodicity, piecewise linear functions and the technique of using functions $|x|$ and $\{x\}$.

Quadratic analogs of $\sin x$ and $\cos x$.

Applying to the function $f_1(x) = |x-1|$ the following chain of transformations
$f_1(x) \longmapsto f_2(x) = -|x-1| = -\left|4 \cdot \dfrac{x+1}{4} - 2\right| \longmapsto f_3(x) = -\left|4\left\{\dfrac{x+1}{4}\right\} - 2\right| \longmapsto$
$f_4(x) = 1 - \left|4\left\{\dfrac{x+1}{4}\right\} - 2\right|$ we obtain 4−periodic function $Sq(x) := f_4(x)$
named Quadratic Sine, that is $Sq(x) = 1 - \left|4\left\{\dfrac{x+1}{4}\right\} - 2\right|$
(see pic.1 and on pic.2 graph colored by red)..
Denoting also
$Cq(x) := Sq(x+1)$ (on pic.2 graph colored by green)
we obtain another 4−periodic function, named Quadratic Cosine.



Pic.1



Pic.2

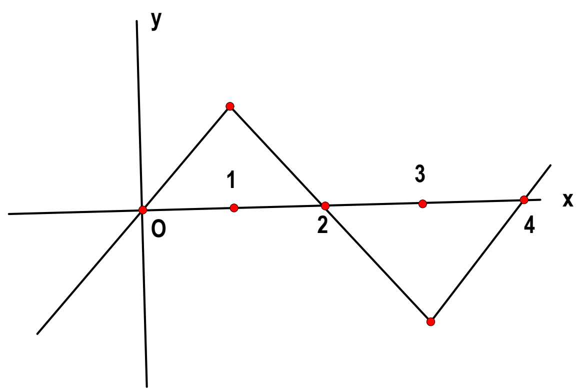
Remark1.
Both function $Sq(x)$, $Cq(x)$ completely defined by its piecewise restrictions on the $[0,4)$:

$$Sq(x)=\begin{cases} x, \text{ if } 0\le x<1 \\ 2-x, \text{ if } 1\le x<3 \\ x-4, \text{ if } 3\le x<4 \end{cases}, \quad Cq(x)=\begin{cases} 1-x, \text{ if } 0\le x<2 \\ x-3, \text{ if } 2\le x<4 \end{cases}.$$

Thus function $Sq(x)$ on $\mathbb{R}$ can be obtained as composition piecewise function

$$h(x)=\begin{cases} x, \text{ if } x<1 \\ 2-x, \text{ if } 1\le x<3 \\ x-4, \text{ if } 3\le x \end{cases} \quad \text{(see pic.3)}$$

with $4-$periodical function $4\left\{\frac{x}{4}\right\}$, namely, $Sq(x):=h\left(4\left\{\frac{x}{4}\right\}\right)$.



Pic..3

For function $h(x)$ we can obtain convenient representation in form
$h(x) = p|x-1| + q|x-3| + ax + b$,where
parameters p, q, a, b can be obtained by consideration $h(x)$ on $(-\infty, 1)$, $[1, 3)$, $[3, \infty)$.
For $x \in (-\infty, 1)$ we have $p(1-x) + q(3-x) + ax + b = x \iff$
$x(-p-q) + p + 3q + b = x \implies \begin{cases} -p-q+a=1 \\ p+3q+b=0 \end{cases}$;
for $x \in [1, 3)$ we have
$p(x-1) + q(3-x) + ax + b = 2 - x \iff \begin{cases} p-q+a=-1 \\ -p+3q+b=2 \end{cases}$;
and for $[3, \infty)$ we have
$p(x-1) + q(x-3) + ax + b = x - 4 \iff \begin{cases} p+q+a=1 \\ -p-3q+b=-4 \end{cases}$.
Adding first and third equations in the system $\begin{cases} -p-q+a=1 \\ p-q+a=-1 \\ p+q+a=1 \end{cases}$
we obtain $a = 1$ and further from
$\begin{cases} -p-q=0 \\ p-q=-2 \end{cases}$ obtain $p = -1, q = 1$.
Substitution p, q in $p + 3q + b = 0$ give us $b = -2$ and easy to see
that obtained values for a, b, p and q satisfy the two remaining equations
$-p + 3q + b = 2$ and $-p - 3q + b = -4$.
Thus, $h(x) = |x-3| - |x-1| + x - 2$ and, therefore, we have

$$Sq(x) = h\left(4\left\{\frac{x}{4}\right\}\right) = \left|4\left\{\frac{x}{4}\right\} - 3\right| - \left|4\left\{\frac{x}{4}\right\} - 1\right| + 4\left\{\frac{x}{4}\right\} - 2.$$

Since $Sq(x) = 1 - \left|4\left\{\frac{x+1}{4}\right\} - 2\right|$ we also get identity

$$\left|4\left\{\frac{x}{4}\right\} - 3\right| - \left|4\left\{\frac{x}{4}\right\} - 1\right| + \left|4\left\{\frac{x+1}{4}\right\} - 2\right| + 4\left\{\frac{x}{4}\right\} = 3.$$

Properties of $Sq(x)$, $Cq(x)$.
1. $Sq(x+2) = -Sq(x)$ and $Cq(x+2) = -Cq(x)$, $Cq(x+1) = -Sq(x)$..

Proof.

Since $Sq(x)$ is $4-$periodic then suffices to prove $Sq(x+2) = -Sq(x)$ for $x \in [0,4)$.

Let $x \in [0,1)$ then $Sq(x) = x$ and $\dfrac{3}{4} \leq \dfrac{x+3}{4} < 1$. Hence, $\left\{\dfrac{x+3}{4}\right\} = \dfrac{x+3}{4} \implies$

$Sq(x+2) = 1 - |x+1| = 1 - x - 1 = -x = Sq(x)$;

Let $x \in [1,4)$ then $Sq(x) = \begin{cases} 2-x, \text{if } x \in [1,3) \\ x = 4, \text{ if } x \in [3,4) \end{cases}$

and since $1 \leq \dfrac{x+3}{4} < \dfrac{7}{2}$ then $\left\{\dfrac{x+3}{4}\right\} = \dfrac{x+3}{4} - 1 = \dfrac{x-1}{4}$ and, therefore,

$Sq(x+2) = 1 - |x-3| = \begin{cases} 1 + x - 3 = x - 2, \text{ if } x \in [1,3) \\ 1 - (x-3) = 4 - x, \text{ if } x \in [3,4) \end{cases} = -Sq(x)$.

Also we have $Cq(x+2) = Sq((x+2)+1) = Sq((x+1)+2) = -Sq((x+1)) = -Cq(x)$

and $Cq(x+1) = Sq((x+1)+1) = Sq(x+2) = -Sq(x)$.

Corollary.

$|Cq(x)|, |Sq(x)|$ are $2-$periodic functions.

Indeed, since $|Cq(x+1)| = |-Sq(x)| = |Sq(x)|$ and $|Sq(x+1)| = |-Sq(x)| = |Sq(x)|$.

2. $|Cq(x)| + |Sq(x)| = 1$.(see pic.3).

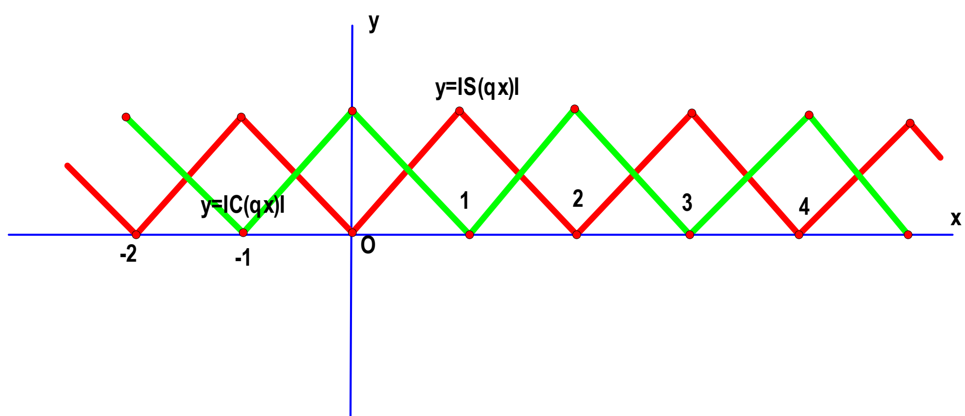
Proof.

First note that $|Cq(x)| + |Sq(x)|$ are $1-$periodic functions. Indeed, since $Cq(x+1) = -Sq(x)$

$Sq(x+1) = Cq(x)$ then $|Cq(x+1)| + |Sq(x+1)| = |-Sq(x)| + |Cq(x)| = |Cq(x)| + |Sq(x)|$.

Thus, suffices to prove identity $|Cq(x)| + |Sq(x)| = 1$ for $x \in [0,1)$.

Let $x \in [0,1)$ then $Cq(x) = 1 - x + x = 1$.



Pic.4

3. $|Cq(x)| - |Sq(x)| = Cq(2x)$.

Proof.

Since $Cq\left(2x\right)$ is $2-$periodic and $Cq\left(2\left(x+1\right)\right)=Cq\left(2x+2\right)=-Cq\left(2x\right)$ and

$\left|Cq\left(x+1\right)\right|-\left|Sq\left(x+1\right)\right|=\left|-Sq\left(x\right)\right|-\left|Cq\left(x\right)\right|=-\left(\left|Cq\left(x\right)\right|-\left|Sq\left(x\right)\right|\right)$ suffices to prove

identity only for $x\in[0,1)$. Let $x\in[0,1)$ then $\left|Cq\left(x\right)\right|-\left|Sq\left(x\right)\right|=1-x-x=1-2x=Cq\left(2x\right)$,

because for $x\in[0,1)$ we have $0\leq 2x<2$ and, therefore, $Cq\left(2x\right)=1-2x$.

4. $1+Cq\left(2x\right)=2\left|Cq\left(x\right)\right|$ and $1-Cq\left(2x\right)=2\left|Sq\left(x\right)\right|$.

Proof.

Adding $\left|Cq\left(x\right)\right|-\left|Sq\left(x\right)\right|=Cq\left(2x\right)$ and $\left|Cq\left(x\right)\right|+\left|Sq\left(x\right)\right|=1$ we obtain $1+Cq\left(2x\right)=2\left|Cq\left(x\right)\right|$

and by subtraction $\left|Cq\left(x\right)\right|-\left|Sq\left(x\right)\right|=Cq\left(2x\right)$ from $\left|Cq\left(x\right)\right|+\left|Sq\left(x\right)\right|=1$ obtain

$1-Cq\left(2x\right)=2\left|Sq\left(x\right)\right|$.

5. For any x,y such that $|x|+|y|=1$ there is only $t\in[0,4)$ such that $x=Cq\left(t\right),y=Sq\left(t\right)$.

(Thus,we obtain mapping $t\longmapsto\left(Cq\left(t\right),Sq\left(t\right)\right):\mathbb{R}\longrightarrow\mathbb{R}^2$ and image of this mapping is "unit" square

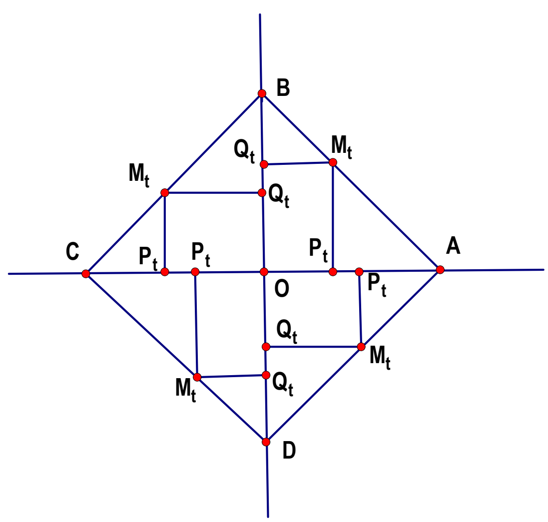
$$\left\{(x,y)\mid x,y\in\mathbb{R}^2,|x|+|y|=1\right\}.$$

which can be considered as a "circle" with radius 1 if we agree to measure the distance

between two points $A\left(x_1,y_1\right)$ and $B\left(x_2,y_2\right)$ using formula $dist\left(A,B\right)=\left|x_2-x_1\right|+\left|y_2-y_1\right|$

instead $dist\left(A,B\right)=\sqrt{\left|x_2-x_1\right|^2+\left|y_2-y_1\right|^2}$.)

Proof.



Pic.5

Accordingly to standard orientation on plain (counterclockwise) for any real t we

set in correspondence point M_t on square such that length of oriented broken line

AM_t equal $\sqrt{2}t$.Since perimeter of square $ABCD$ equal $\sqrt{2}t$ then suffices consider

$t \in [0,4)$. Let $P_t(x_t,0), Q_t(0,y_t)$ is orthogonal projections of M_t on axes OX and OY

respectively.

Let $t \in [0,1)$ then $M_t(x_t,y_t) \in AB$. Since $|AP_t| = |OQ_t| = t$ then $x_t = 1 - t = Cq(t)$ and

$y_t = t = Sq(t)$;

Let $t \in [1,2)$ then $M_t(x_t,y_t) \in BC$. Since $|BM_t| = \sqrt{2}(t-1)$ then

$|BQ_t| = |MQ_t| = |OP_t| = t - 1$ and, therefore, $x_t = -(t-1) = 1 - t = Cq(t)$ and

$y_t = 1 - (t-1) = 2 - t = Sq(t)$;

Let $t \in [2,3)$ then $M_t(x_t,y_t) \in CD$. Since $|CM_t| = \sqrt{2}(t-2)$ then $|CP_t| = |M_tP_t| =$

$|OQ_t| = t - 2$ and $|OP_t| = 1 - (t-2) = 3 - t$. Therefore, $x_t = -(3-t) = t - 3 = Cq(t)$

and $y_t = -(t-2) = 2 - t = Sq(t)$;

Let $t \in [3,4)$ then $M_t(x_t,y_t) \in DA$.

$y_t = 1-(t-1) = 2-t = Sq(t)$; $|DM_t| = \sqrt{2}(t-3)$ then $|DQ_t| = |MQ_t| = |OP_t| = t - 3$

and $|OQ_t| = 1 - (t-3) = 4 - t$.Therefore, $x_t = t - 3 = Cq(t)$ and $y_t = -(4-t) = t - 4$.

Thus $\left\{(x,y) \mid x,y \in \mathbb{R}^2, |x| + |y| = 1\right\} = \{(Cq(t), Sq(t)) \mid t \in [0,4)\} =$

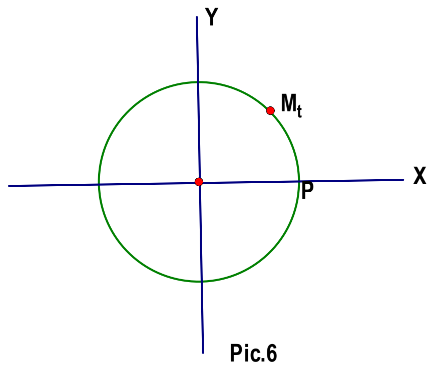
$\{(Cq(t), Sq(t)) \mid t \in \mathbb{R}\}$.

Remark 2.

We can see analogy between Quadratic cosine and sine $Cq(t)$, $Sq(t)$ with trigonometric

functions $\cos\dfrac{\pi t}{2}$ and $\sin\dfrac{\pi t}{2}$ as coordinates of point M_t ,on the unit circle such that

oriented length of arc PM_t equal $\dfrac{\pi t}{2}$, where $t \in [0,4)$.



Pic.6